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C-Scattered Fuzzy Topological Spaces

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Abstract—In this paper, we define the concept of C-scattered fuzzy topological spaces and obtain some properties about them. In particular, we study the relation between C-scattered spaces and its fuzzy extension, it is proved that C-scattered fuzzy topological spaces are invariant by fuzzy perfect maps, and that, in the realm of paracompact fuzzy topological spaces, the C-scattered spaces verify that their product by other fuzzy spaces is also paracompact fuzzy.

Keywords—Topology, C-scattered and scattered topological spaces, Compactness, Fuzzy perfect maps, S-paracompactness, S^*-paracompactness, Fuzzy paracompactness, $*$-Fuzzy paracompactness.

1. INTRODUCTION

The concept of C-scattered topological space has been defined by Telgársky [1]. A space X is said to be C-scattered if each of its nonempty closed subspaces contains a compact set with nonempty relative interior. Then, it is a simultaneous generalization of scattered and of locally compact spaces, and it is interesting because this class has good properties in relation with the product of paracompact spaces. See the papers of Telgársky [1–4], Ormosadze [5], Dodon and Čoban [6,7], Friedler, Martin and Williams [8], Nogura [9], Yajima [10] and the author [11–13].

In this paper, we define for fuzzy topological spaces in the Chang's sense, the notion of C-scattered (and scattered) spaces, and obtain various properties of these spaces. In particular, the relationship between C-scattered spaces and their fuzzy extensions is studied, it is proved that the image by a fuzzy perfect map of all C-scattered fuzzy topological space, is also C-scattered, and that, in the realm of paracompact fuzzy topological spaces (in the various senses), the C-scattered spaces verify that its product by other fuzzy spaces is also paracompact fuzzy.

The standard definitions on fuzzy topology are in [14,15], moreover, in this paper we will use the following known notions.

DEFINITION 1. *(See [16].) Let (X,τ) be a topological space and $\omega(\tau)$ be the set of all lower semicontinuous functions from (X,τ) to the unit interval equipped with the usual topology, then $(X,\omega(\tau))$ is called the weakly induced fuzzy topological space by (X,τ).*

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DEFINITION 2. *(See [17].) A map f from a fuzzy topological space X to a fuzzy topological space Y is called fuzzy perfect if f is onto, fuzzy closed, F-continuous and $f^{-1}(y_\alpha)$ is compact for each fuzzy point y_α in Y.*

DEFINITION 3. *(See [18].) Let $r \in (0,1]$. A fuzzy topological space X is called r-paracompact (respectively, r^*-paracompact) if for each r-open Q-cover of X, there exists an open refinement of it which is both locally finite (respectively, $*$-locally finite) in X and an $r-Q$-cover of X. The fuzzy topological space X is called S-paracompact (respectively, S^*-paracompact) if for every $r \in (0,1]$, X is r-paracompact (respectively, r^*-paracompact).*

DEFINITION 4. *(See [19].) Let μ be a fuzzy set in a fuzzy topological space X. We say that μ is fuzzy paracompact (respectively, $*$-fuzzy paracompact) if for each open cover in the Lowen's sense $\mathcal{U}$ of μ, and for each $r \in (0,1]$, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is both locally finite (respectively, $*$-locally finite) in μ and cover of $\mu - r$ in the Lowen's sense. We say that a fuzzy topological space X is fuzzy paracompact (respectively, $*$-fuzzy paracompact) if each constant fuzzy set in X is fuzzy paracompact (respectively, $*$-fuzzy paracompact).*

2. C-SCATTERED FUZZY TOPOLOGICAL SPACES

DEFINITION 5. *A fuzzy topological space X will be called C-scattered fuzzy (respectively, scattered fuzzy) if for every closed fuzzy set μ of X, there exists a fuzzy point $x_\lambda \leq \mu$, and there exists a fuzzy neighborhood ν of x_λ such that $\nu \wedge \mu$ is compact fuzzy (respectively, fuzzy point) in X.*

LEMMA 2.1. *Let (X,τ) be a topological space and μ be a fuzzy set in X. If μ is fuzzy compact, then $\mu^{-1}((0,1])$ is compact.*

PROOF. Let $\{U_j\}_{j\in J}$ be an open covering of $\mu^{-1}((0,1])$, then $\mu^{-1}((0,1]) = \bigcup_{j\in J} U_j$, thus for each $z \in X$, either $0 \leq (\bigvee_{j\in J} \chi_{U_j})(z)$ if $\mu(z) = 0$, or there is $j_0 \in J$ such that $z \in U_{j_0}$, then $\chi_{U_{j_0}}(z) = 1$ and $\mu(z) \leq (\bigvee_{j\in J} \chi_{U_j})(z)$. So, $\mu \leq \bigvee_{j\in J} \chi_{U_j}$ and there is some finite subset J_0 in J such that $\mu \leq \bigvee_{j\in J_0} \chi_{U_j}$. If $\mu(z) \neq 0$, we have that $0 < \mu(z) \leq (\bigvee_{j\in J_0} \chi_{U_j})(z) = \chi_{\bigcup_{j\in J} U_j}(z)$ and $\mu^{-1}((0,1]) \subset \bigcup_{j\in J_0} U_j \subset \mu^{-1}((0,1])$.

PROPOSITION 2.2. *If $(X,\omega(\tau))$ is C-scattered fuzzy, then (X,τ) is a C-scattered topological space.*

PROOF. For every closed set C of (X,τ), we have $X\backslash C \in \tau$, then $\chi_{X\backslash C}$ is open fuzzy and χ_C is closed fuzzy in $(X,\omega(\tau))$. From the hypothesis, there exists a fuzzy point $x_\lambda \leq \chi_C$ (then $\lambda \leq \chi_C(x)$ and $x \in C$), and there exists a fuzzy neighborhood ν of x_λ such that $\nu \wedge \chi_C$ is fuzzy compact. Thus, there is an open fuzzy ν^* such that $x_\lambda \leq \nu^* \leq \nu$. So, $\nu^{*-1}((a,1])$ is open in (X,τ), for all $a \in [0,1)$ and $\nu^{*-1}((a.1]) \subset \nu^{-1}((a,1])$, moreover, $\lambda \leq \nu^*(x)$, then $x \in \nu^{*-1}([\lambda,1]) \subset \nu^{*-1}((a,1])$, for all $a \in [0,\lambda)$ and $x \in \nu^{*-1}((0,1]) \subset \nu^{-1}((0,1])$. Also, $(\nu \wedge \chi_C)^{-1}((a,1]) = \{z \mid \min\{\nu(z),\chi_C(z)\} > a\} = \{z \in C \mid \nu(z) > a\} = C \cap \nu^{-1}((a,1])$, for all $a \in [0,1)$.

And finally, $C \cap \nu^{-1}((0,1])$ is compact in C by the lemma.

REMARK 2.3. The converse of the above proposition is not true.

PROOF. Let (X,τ) be a discrete topological space, then $\omega(\tau) = \{\mu : X \to [0,1] \text{ map}\}$. The identically zero map μ_0 is closed fuzzy in $(X,\omega(\tau))$ and there is no fuzzy point x_λ in X such that $x_\lambda \leq \mu_0$. Thus, $(X,\omega(\tau))$ is not C-scattered fuzzy.

PROPOSITION 2.4. *If $(X,\omega(\tau))$ is scattered fuzzy, then (X,τ) is a scattered topological space.*

PROOF. For every closed C of (X,τ), we have that χ_C is closed fuzzy. By the hypothesis, there exists a fuzzy point $x_\lambda \leq \chi_C$ and a fuzzy neighborhood ν of x_λ such that $\nu \wedge \chi_C$ is x_λ. Analogously to the above proposition, there is an open fuzzy ν^* such that $x_\lambda \leq \nu^* \leq \nu$. Then $\nu^{-1}((0,1])$ is a neighborhood of x in (X,τ) and $\{x\} = \operatorname{supp}(\nu \wedge \chi_C) = \{z \mid \min\{\nu(z), \chi_C(z)\} > 0\} = \{z \mid z \in C,\ \nu(z) > 0\} = C \cap \nu^{-1}((0,1])$.

REMARK 2.5. The converse of the last proposition in not true.

(See the counterexample for C-scattered spaces.)

PROPOSITION 2.6. *Let X and Y be two fuzzy topological spaces and $f : X \to Y$ be a fuzzy perfect map. If X is C-scattered fuzzy, then Y is also C-scattered.*

PROOF. For each closed fuzzy set μ in Y, $f^{-1}(\mu)$ is closed fuzzy in X, then, by the hypothesis, there is a fuzzy point $x_\lambda \le f^{-1}(\mu)$ and a fuzzy neighborhood ν of x_λ in X such that $\nu \wedge f^{-1}(\mu)$ is compact fuzzy.

But, $x_\lambda \le f^{-1}(\mu)$ is equivalent to $\lambda \le f^{-1}(\mu)(x) = \mu(f(x))$ and this implies that $f(x_\lambda) \le \mu$, because

$$f(x_\lambda)(y) = \sup_{z \in f^{-1}(y)} \{x_\lambda(z)\} = \begin{cases} \lambda, & \text{if } y = f(x), \\ 0, & \text{if } y \ne f(x), \end{cases}$$

then $f(x_\lambda)$ is the fuzzy point of support $f(x)$ and value λ.

Now, Y is a quotient fuzzy topological space, then $f(x_\lambda) \le f(\nu)$ that is a fuzzy neighborhood because $x_\lambda \le \nu \le f^{-1}(f(\nu))$ and f is a fuzzy identification [20].

Finally, for each $y \in Y$,

$$\begin{aligned} f\left(\nu \wedge f^{-1}(\mu)\right)(y) &= \sup_{z \in f^{-1}(y)} \left\{\left(\nu \wedge f^{-1}(\mu)\right)(z)\right\} = \sup_{z \in f^{-1}(y)} \{\min\{\nu(z), \mu(f(z))\}\} \\ &= \sup_{z \in f^{-1}(y)} \{\min\{\nu(z), \mu(y)\}\} \text{ and} \end{aligned}$$

$$(f(\nu) \wedge \mu)(y) = \min\{f(\nu(y)), \mu(y)\} = \min\left\{\sup_{z \in f^{-1}(y)} \{\nu(z)\},\ \mu(y)\right\},$$

and this is equal in the three cases:

(i) $\nu(z) < \mu(y)$ for all z,
(ii) $\nu(z) \ge \mu(y)$ for all z,
(iii) there are $z_0, z_1 \in f^{-1}(y)$ with $\nu(z_0) \ge \mu(y) > \nu(z_1)$.

So, $f(\nu \wedge f^{-1}(\mu)) = f(\nu) \wedge \mu$ and this is a compact fuzzy [14, Theorem 5.2] and a fuzzy neighborhood of $f(x_\lambda)$ in μ.

PROPOSITION 2.7. *Let X be a C-scattered fuzzy and S-paracompact fuzzy topological space (respectively, S^*-paracompact, fuzzy paracompact, or $*$-fuzzy paracompact), then $X \times Y$ is S-paracompact (respectively, the other kinds of fuzzy paracompactness) if and only if Y is this.*

PROOF. If $(X, \omega(\tau))$ is C-scattered fuzzy and S-paracompact (analogously for the other kinds of fuzzy paracompactness), then (X,τ) is C-scattered (by Proposition 2.2) and paracompact space [18,19].

Thus, $(X,\omega(\tau)) \times (Y, \omega(s))$ is S-paracompact if and only if $(X,\tau) \times (Y,s)$ is paracompact, and this is equivalent to (Y,s) paracompact [1], and $(Y,\omega(s))$ S-paracompact.

PROPOSITION 2.8. *Let X and Y be two fuzzy topological spaces and $f : X \to Y$ be a fuzzy perfect map. Then, if μ is a compact fuzzy set of Y, so is $f^{-1}(\mu)$ in X.*

PROOF. Let $\{\mu_j \mid j \in J\}$ be a family of open fuzzy sets of X such that $f^{-1}(\mu) \le \bigvee_{j\in J} \mu_j$. We will denote as J_0 the set of all finite subsets of J, and $\mu_F = \bigvee_{j \in F} \mu_j$ for every $F \in J_0$.

For each fuzzy point $x_\lambda \le \mu$, we have that $f^{-1}(x_\lambda)$ is compact fuzzy, then there exists $F \in J_0$ such that $f^{-1}(x_\lambda) \le \mu_F$, thus, $\lambda \le \mu_F(t)$ for all $t \in f^{-1}(x)$, $\lambda \le \inf_{t \in f^{-1}(x)}\{\mu_F(t)\} = (f(\mu'_F))'(x)$, hence, $x_\lambda \le (f(\mu'_F))'$ and this is an open fuzzy set in Y. Then, if we collect the set $\{F_\lambda \in J_0 \mid \mu(x) = \lambda, x \in X\}$, we have an open fuzzy covering for μ, and there exist $F_1, \dots, F_k \in J_0$ such that $\mu \leq \bigvee_{i=1}^k (f((\mu'_{F_i}))'$. Then

$$f^{-1}(\mu) \leq \bigvee_{i=1}^{k} f^{-1}\left(\left(f\left(\mu'_{F_i}\right)\right)'\right) \leq \bigvee_{i=1}^{k} \mu_{F_i} = \bigvee_{j \in \bigcup_{i=1}^k F_i} \mu_j$$

(because $f^{-1}((f(\mu'_F))')(z) = (f(\mu'_F))'(f(z)) = 1 - f(\mu'_F)(f(z)) = 1 - \sup_{t \in f^{-1}(f(z))}\{\mu'_F(t)\} = \inf_{t \in f^{-1}(f(z))}\{\mu_F(t)\} \leq \mu_F(z)$). Thus, $f^{-1}(\mu)$ is compact fuzzy.

Open Problem

Let X and Y be two fuzzy topological spaces and $f : X \to Y$ be a fuzzy perfect map. If Y is C-scattered fuzzy, is X C-scattered fuzzy also?

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